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Zeros of the Partition Function and Gaussian Inequalities for the Plane Rotator Model

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The partition function for ferromagnetic plane rotators in a complex external field $\boldsymbol{\mu}$, with $|\mathrm{Im}\,\boldsymbol{\mu}| \leqslant |\mathrm{Re}\,\boldsymbol{\mu}|$, is bounded below in modulus by its value at $\boldsymbol{\mu} = 0$. The proof is based on complex combinations of duplicated variables which are positive definite on a subgroup of the configuration group. In the isotropic situation (and $\boldsymbol{\mu} = 0$), the associated "Gaussian inequalities" imply that all truncated correlation functions decay at least as the two-point function.

KEY WORDS: Plane rotators; Lee–Yang zeros; Gaussian inequalities.

1. INTRODUCTION

We consider a family of plane rotators, i.e., a family of random two-dimensional unit vectors $\{\mathbf{S}_j = (S_j^1, S_j^2): j = 1,..., N\}$ with joint probability distribution on $(\mathbb{R}^2)^N$:

$$Z_N^{-1} \exp\left\{ \sum_{j,k=1}^{N} (J_{jk}^1 S_j^1 S_k^1 + J_{jk}^2 S_j^2 S_k^2) + \sum_{j=1}^{N} \boldsymbol{\mu}_j \cdot \mathbf{S}_j \right\} \prod_{j=1}^{N} \delta(\mathbf{S}_j^2 - 1)\, d\mathbf{S}_j \quad (1)$$

where

$$\begin{aligned} Z_N &= Z_N((\boldsymbol{\mu}_j)_{j=1,\ldots,N}) \\ &= \int \exp\left\{ \sum_{j,k=1}^{N} (J_{jk}^1 S_j^1 S_k^1 + J_{jk}^2 S_j^2 S_k^2) + \sum_{j=1}^{N} \boldsymbol{\mu}_j \cdot \mathbf{S}_j \right\} \prod_{j=1}^{N} \delta(\mathbf{S}_j^2 - 1)\, d\mathbf{S}_j \end{aligned} \quad (2)$$

Models of physical interest may be obtained in a thermodynamic limit $N \to \infty$, in situations where the pressure can be defined,

$$p((\boldsymbol{\mu}_j)_j) = \lim_{N\to\infty} N^{-1} \log Z_N((\boldsymbol{\mu}_j)_{j=1,\ldots,N})$$

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and depends only on bulk variables: asymptotically, it varies with $(\mu_j)_j$ only when a finite fraction (i.e., proportional to N) of the μ_j are varied. In particular, the pressure is independent of "boundary conditions." If, moreover, we can replace μ_j by $\mu_j + \zeta$ for an arbitrary fraction of the sites j and have the pressure analytic in ζ around $\zeta = 0$, it follows that the correlation functions averaged over this arbitrary fraction are also independent of boundary conditions. All these partial averages presumably determine the state of the system, which would then be unique. This motivates the study of the analyticity of the pressure as a function of a nonuniform external field $(\mu_j)_j$. The problem is to find complex neighborhoods of $(\mu_j)_{j=1,\ldots,N}$ which do not shrink to the reals as $N \to \infty$, where the partition function Z_N has no zeros.

This program was initiated by Lee and Yang[9] for the Ising model and was continued for quantum spins by Asano[1] and Suzuki and Fisher.[12] This allowed a complicated approximation[5] to the plane rotator model which implied analyticity of $\log Z_N((\mu_j)_{j=1,\ldots,N})$ in the variables $(\mu_j^1)_{j=1,\ldots,N}$ only, in the region

$$\operatorname{Re} \mu_j^1 > 0, \qquad j = 1,\ldots, N$$

provided

$$|J_{jk}^2| \leqslant J_{jk}^1, \qquad j, k = 1,\ldots, N; \qquad \mu_j^2 \in \mathbb{R}, \quad j = 1,\ldots, N$$

Using full analyticity in the large external field region, Fröhlich[7] then proved analyticity in $(\mu_j^1)_j$ *and* $(\mu_j^2)_j$ in some complex neighborhood of

$$\mu_j^1 > 0, \qquad \mu_j^2 \in \mathbb{R} \qquad \forall j \tag{3}$$

and also indicated a precise procedure to implement the program outlined above (unicity of the state in the same region).

The present paper is first devoted to more direct proofs of analyticity for the plane rotator model: without any approximation procedure, and without appealing to large external field expansions, we shall prove a stronger result, namely

$$|Z_N((\mu_j)_{j=1,\ldots,N})| \geqslant Z_N(0) \tag{4}$$

for $(\mu_j)_{j=1,\ldots,N}$ belonging to an explicitly constructed complex region (Theorems 1 and 2 below). The intersection of this region with the reals is, however, a little smaller than (3).

Our second topic is the "Gaussian inequality" for the same model, but in the absence of an external field. Gaussian inequalities were first proven for Ising spins by Newman[10] as a consequence[2] of the Lee and Yang theorem. In our framework, the same tools will provide at the same time the lower bound (4) and a strong version of the Gaussian inequality, applicable to

[2] This concerns a weak form of the inequality. Newman's stronger results are based on combinatorial methods.

truncated correlations, such as that given by Bricmont.[2] These tools generalize previous work[3] on one-component classical ferromagnets: appropriate complex combinations of duplicated spin variables are shown to be positive definite on (a subgroup of) the configuration group. All our results express the fact that such combinations have a positive integral over the configuration space.

If the lengths of the rotators are also random variables, the method does not seem to be directly applicable and one has to appeal to a random walk approximation.[5]

2. A LOWER BOUND ON THE MODULUS OF THE COMPLEX PARTITION FUNCTION

Our method is restricted to two-body ferromagnetic couplings. To simplify the formulation, we first consider the isotropic case:

Theorem 1. Let $Z_N((\mu_j)_j)$ be given by (2) with

$$J^1_{jk} = J^2_{jk} \geqslant 0, \qquad j, k = 1,..., N$$

Let

$$\mathbf{x}_j = \operatorname{Re} \mu_j, \qquad \mathbf{y}_j = \operatorname{Im} \mu_j, \qquad j = 1,..., N$$

Let $\mathbf{y}_j^\perp$ be the image of $\mathbf{y}_j$ under a rotation by $\pi/2$. Suppose that

$$(\mathbf{x}_j \pm \mathbf{y}_j^\perp)\cdot(\mathbf{x}_k \pm \mathbf{y}_k^\perp) \geqslant 0, \qquad j, k = 1,..., N \tag{5}$$

Then

$$|Z_N((\mu_j)_j)| \geqslant Z_N(0) \tag{6}$$

Remarks. Theorem 1 implies analyticity of the pressure in a neighborhood of a real external field $(\mathbf{x}_j)_j$ provided

$$\inf_{j,k} \mathbf{x}_j\cdot\mathbf{x}_k > 0$$

When all the real external fields $\mathbf{x}_j$ point in the same direction, condition (5) reads

$$|\mathbf{y}_j| \leqslant |\mathbf{x}_j| \qquad \forall_j$$

Proof. We introduce an independent copy $(\mathbf{S}_j')_{j=1,...,N}$ with the same probability distribution (1) for real $(\mu_j)_j$. When extending the partition function to complex external fields, we use $(\mu_j)_j$ for the original copy, and the complex conjugate $(\mu_j^*)_j$ for the primed copy. We then have

$$\begin{aligned}|Z_N((\mu_j)_j)|^2 &= Z_N((\mu_j)_j) Z_N((\mu_j^*)_j) \\ &= \int \exp\left\{ \sum_{j,k=1}^{N} J^1_{jk}(\mathbf{S}_j\cdot\mathbf{S}_k + \mathbf{S}_j'\cdot\mathbf{S}_k') + \sum_{j=1}^{N} (\mu_j\cdot\mathbf{S}_j + \mu_j^*\cdot\mathbf{S}_j') \right\} \\ &\quad \times \prod_{j=1}^{N} \delta(\mathbf{S}_j^2 - 1)\, d\mathbf{S}_j\, \delta(\mathbf{S}_j'^2 - 1)\, d\mathbf{S}_j' \end{aligned} \tag{7}$$

Clearly, the a priori measure is invariant under the action of $(O(2) \times O(2))^N$. We shall choose a particular subgroup and express the integral (7) as the sum of the integrals over the orbits associated with the action of this subgroup. In order to have a positive integral over each orbit, we choose the subgroup so that the ferromagnetic interaction and the complex linear part are both positive definite on it, for each orbit. More trivially, we make the following change of variables:

$$\begin{aligned} \frac{\mathbf{S}_j + \mathbf{S}_j'}{2} &= (\cos \alpha_j)\hat{\mathbf{t}}_j = (\cos \alpha_j)\begin{pmatrix} \cos \beta_j \\ \sin \beta_j \end{pmatrix}, \qquad j = 1,\ldots, N \\ \frac{\mathbf{S}_j - \mathbf{S}_j'}{2} &= (\sin \alpha_j)\hat{\mathbf{t}}_j^{\perp} = (\sin \alpha_j)\begin{pmatrix} \sin \beta_j \\ -\cos \beta_j \end{pmatrix}, \qquad j = 1,\ldots, N \end{aligned} \tag{8}$$

where $\hat{\mathbf{t}}_j$ and $\hat{\mathbf{t}}_j^{\perp}$ are orthogonal unit vectors associated with angles β_j and $\beta_j - \frac{1}{2}\pi$, respectively. Rotation invariance gives the new a priori measure

$$\delta(\mathbf{S}_j^2 - 1)\, d\mathbf{S}_j \cdot \delta(\mathbf{S}_j'^2 - 1)\, d\mathbf{S}_j' \to d\alpha_j\, d\beta_j$$

The direction $\beta_j = 0$ will now be chosen, the same for all j, but depending on the external fields:

$$\begin{aligned} \boldsymbol{\mu}_j \cdot \mathbf{S}_j + \boldsymbol{\mu}_j^* \cdot \mathbf{S}_j' &= 2(\cos \alpha_j)\mathbf{x}_j \cdot \hat{\mathbf{t}}_j + 2i(\sin \alpha_j)\mathbf{y}_j \cdot \hat{\mathbf{t}}_j^{\perp} \\ &= 2(\cos \alpha_j)\mathbf{x}_j \cdot \hat{\mathbf{t}}_j + 2i(\sin \alpha_j)\mathbf{y}_j^{\perp} \cdot \hat{\mathbf{t}}_j \\ &= e^{i\alpha_j}(\mathbf{x}_j + \mathbf{y}_j^{\perp}) \cdot \hat{\mathbf{t}}_j + e^{-i\alpha_j}(\mathbf{x}_j - \mathbf{y}_j^{\perp}) \cdot \hat{\mathbf{t}}_j \end{aligned}$$

Under hypothesis (5), all the $(\mathbf{x}_j \pm \mathbf{y}_j^{\perp})$, $j = 1,\ldots, N$, lie in a given quadrant, so that we may choose the origin $\beta = 0$ in such a way that

$$\begin{aligned} (\mathbf{x}_j + \mathbf{y}_j^{\perp}) \cdot \hat{\mathbf{t}}_j &= a_j \cos \beta_j + b_j \sin \beta_j \\ (\mathbf{x}_j - \mathbf{y}_j^{\perp}) \cdot \hat{\mathbf{t}}_j &= c_j \cos \beta_j + d_j \sin \beta_j \end{aligned}$$

with

$$a_j \geqslant 0, \qquad b_j \geqslant 0, \qquad c_j \geqslant 0, \qquad d_j \geqslant 0; \qquad j = 1,\ldots, N$$

In any case, the *isotropic* coupling terms will read

$$\begin{aligned} J_{jk}^1(\mathbf{S}_j \cdot \mathbf{S}_k + \mathbf{S}_j' \cdot \mathbf{S}_k') = J_{jk}^1(&e^{i\alpha_j}e^{-i\alpha_k} + e^{-i\alpha_j}e^{i\alpha_k}) \\ &\times (\cos \beta_j \cos \beta_k + \sin \beta_j \sin \beta_k) \end{aligned}$$

Let us now write

$$\begin{aligned} \cos \beta_j &= \sigma_j |\cos \beta_j|, \qquad \sigma_j = \pm 1 \in \mathbb{Z}_2; \qquad j = 1,\ldots, N \\ \sin \beta_j &= \tau_j |\sin \beta_j|, \qquad \tau_j = \pm 1 \in \mathbb{Z}_2; \qquad j = 1,\ldots, N \end{aligned} \tag{9}$$

and look at orbits in configuration space obtained by the action of $(U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2)^N$ on the variable $(e^{i\alpha_j}, \sigma_j, \tau_j)_{j=1,\ldots,N}$. The integral (7) restricted to every such trajectory is seen to be positive, being the integral of a positive definite function. The bound (6) is obtained by expanding the exponential of the external field and retaining only the first term: the others are also positive definite and therefore give a positive contribution to the integral. The sum over trajectories (i.e., over $|\cos \beta_j|$, $|\sin \beta_j|$, $j = 1,..., N$) clearly does not affect the argument.

We shall now extend Theorem 1 to include the anisotropic case $J^1_{jk} \neq J^2_{jk}$. Notice that the two-body interaction in (1) does *not* contain terms of the form

$$J^{1,2}_{jk}(S_j{}^1 S_k{}^2 + S_j{}^2 S_k{}^1)$$

If such terms were present, there would be for each bond (j, k) a unique change of axes which eliminates these crossed terms and leads to

$$J^1_{jk} \geqslant |J^2_{jk}|$$

In the next theorem, we have to assume that the same choice of axes for all bonds leads to this situation. In other words, the two-body interaction favors the same direction throughout the system. The real external fields will then be allowed to vary within $\pi/4$ from this direction.

Theorem 2. Let $Z_N((\boldsymbol{\mu}_j)_j)$ be given by (2) with

$$J^1_{jk} \geqslant |J^2_{jk}|, \qquad j, k = 1,..., N$$

Let $\mathbf{x}_j$, $\mathbf{y}_j$, $\mathbf{y}_j{}^\perp$ be defined as in Theorem 1. Let $\hat{\mathbf{J}}$ be the direction labeled by 1 in (2) and let $\hat{\mathbf{J}}^\perp$ be its image under a rotation by $\pi/2$. Suppose that

$$(\mathbf{x}_j \pm \mathbf{y}_j{}^\perp)\cdot(\hat{\mathbf{J}} \pm \hat{\mathbf{J}}^\perp) \geqslant 0, \qquad j = 1,..., N \tag{10}$$

Then

$$|Z_N((\boldsymbol{\mu}_j)_j)| \geqslant Z_N(0)$$

Proof. The proof is the same as that of Theorem 1, except that the choice of the origin $\beta = 0$ is imposed by the anisotropy: the two-body interaction will be positive definite on $(U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2)^N$ if and only if $\beta = \pi/4$ corresponds to the dominant ferromagnetic direction $\hat{\mathbf{J}}$. We leave this easy calculation to the reader.

Remark. As for one-component systems,[3] it is possible to prove lower bounds also for correlation functions in a complex external field. The result is simple only for one or two spins:

$$\mathrm{Re}\langle S_j{}^1 \pm S_j{}^2\rangle \geqslant |\mathrm{Im}\langle S_j{}^1 \mp S_j{}^2\rangle| \tag{11}$$

$$\mathrm{Re}\langle S_j{}^1 S_k{}^1 \pm S_j{}^2 S_k{}^2\rangle \geqslant 0 \tag{12}$$

We cannot, however, bound such correlations by zero-field correlations, as we did for the partition function.

3. THE GAUSSIAN INEQUALITY

In the preceding section, we have given tools to prove analyticity in the "positive" external field region. The present section is mainly devoted to the case of zero external field: the situation is very different, but the same tools will provide a straightforward proof of a strong version of the "Gaussian inequality" (see Theorem 3 below). Note that this inequality has also been proven by Bricmont,[2] using the negative correlations of different components of a D-component spin ($D = 1, 2, 3, 4$). The case $D = 1$ is there considered as a special case of $D = 2$ through a duplication. Results for $D = 1$ were previously obtained by Lebowitz,[8] Feldman,[6] Spencer,[11] and Newman.[10] In our framework, $D = 1$ is actually easier than $D = 2$, the conditions on the individual spin distributions are those given in Ref. 3, and the results can be read from Theorem 3 below by suppressing the upper index 1. On the other hand, our method is at present not applicable to $D = 3, 4$.

For simplicity, we first give a theorem ($D = 2$) for the first component only:

Theorem 3. Under the hypotheses of Theorem 2, for any family of elements in $\{1,..., N\}$ indexed by a finite set A, let

$$\begin{aligned}\langle S_A{}^1\rangle &= \Big\langle \prod_{a\in A} S^1_{k_a}\Big\rangle \\ &= Z((\boldsymbol{\mu}_j)_j)^{-1}\int\Big(\prod_{a\in A} S^1_{k_a}\Big)\exp\Big\{\sum_{j,k=1}^{N}(J^1_{jk}S_j{}^1S_k{}^1 + J^2_{jk}S_j{}^2S_k{}^2) \\ &\quad + \sum_{j=1}^{N}\boldsymbol{\mu}_j\cdot\mathbf{S}_j\Big\}\prod_{j=1}^{N}\delta(\mathbf{S}_j{}^2 - 1)\,d\mathbf{S}_j \end{aligned}\tag{13}$$

Suppose in addition that $(\boldsymbol{\mu}_j)_j$ is real. Then:

(i) If $|A| = 2n$, $n \geqslant 2$,

$$\langle S_A{}^1\rangle \leqslant (2^n - 2)^{-1}\sum_{\substack{B\subset A\\ B\neq\varnothing, A\\ |B|\,\text{even}}}\langle S_B{}^1\rangle\langle S_{\tilde{B}}{}^1\rangle \leqslant \sum_{\pi\in\mathscr{P}(A)}\prod_{\{j_1,j_2\}\in\pi}\langle S^1_{j_1}S^1_{j_2}\rangle \tag{14}$$

where $\tilde{B} = A\backslash B$ and $\mathscr{P}(A)$ is the set of partitions of A into pairs ("pairings").

(ii) If $|A_1| = 2n_1$ and $|A_2| = 2n_2$,

$$\begin{aligned}\langle S^1_{A_1}S^1_{A_2}\rangle - \langle S^1_{A_1}\rangle\langle S^1_{A_2}\rangle &\leqslant 2^{-n_1-n_2+1}\sum_{\substack{B_1\subset A_1; B_2\subset A_2\\ |B_1|,|B_2|\,\text{odd}}}\langle S^1_{B_1}S^1_{B_2}\rangle\langle S^1_{\tilde{B}_1}S^1_{\tilde{B}_2}\rangle \\ &\leqslant \sum_{\pi\in\mathscr{P}(A_1;A_2)}\prod_{\{j_1,j_2\}\in\pi}\langle S^1_{j_1}S^1_{j_2}\rangle \end{aligned}\tag{15}$$

where $\tilde{B}_1 = A_1\backslash B_1$, $\tilde{B}_2 = A_2\backslash B_2$, and $\mathscr{P}(A_1; A_2)$ is the set of pairings of $A_1 \cup A_2$ where at least one pair intersects both A_1 and A_2.

Remarks. For isotropic interactions ($J_{jk}^1 = J_{jk}^2$), the direction 1 may be changed, depending on the external field $(\mu_j)_j$. If it vanishes, $(\mu_j)_j \equiv 0$, the direction 1 becomes arbitrary and the inequalities above will hold for any choice of the first component.

When $(\mu_j)_j \equiv 0$, Theorem 3 implies that the truncated correlations of the first component decay at least as the two-point function of this component. Using inequalities[4] between truncated correlations where the choice of components is varied, the same result follows for all truncated correlations.

Proof. The second inequality in (i) can be obtained from the first by induction over n, as was shown by Bricmont. The second inequality in (ii) follows from the first and from (i) similarly. And the first part of (i) and (ii) will follow from a lemma:

Lemma 1. Let $(J_{jk}^1, J_{jk}^2)_{j,k=1,\ldots,N}$ and $(\mu_j)_{j=1,\ldots,N}$ be as in Theorem 2. Consider the linear functional on multinomials in $(S_j{}^1)_{j=1,\ldots,N}$ and $(S_j'{}^1)_{j=1,\ldots,N}$ defined by (13) and

$$\langle S_B{}^1 S_C'{}^1\rangle = \langle S_B{}^1\rangle\overline{\langle S_C{}^1\rangle} \tag{16}$$

Then, for any function ϵ from the index set A to $\{+1, -1\}$, we have

$$\left\langle\prod_{a\in A} (e^{i\epsilon_a\pi/4}S_{k_a}^1 + e^{-i\epsilon_a\pi/4}S_{k_a}'^1)\right\rangle \geqslant 0 \tag{17}$$

Proof of the Lemma. As in the proof of Theorem 1, where $\beta = \pi/4$ is now the direction labeled by 1, we write

$$\begin{aligned}
&[\exp(i\epsilon_a\pi/4)]S_{k_a}^1 + [\exp(-i\epsilon_a\pi/4)]S_{k_a}'^1\\
&\quad = 2^{-1/2}[(S_{k_a}^1 + S_{k_a}'^1) + i\epsilon_a(S_{k_a}^1 - S_{k_a}'^1)]\\
&\quad = 2^{-1}[(\cos\alpha_{k_a})(\cos\beta_{k_a} + \sin\beta_{k_a}) + i\epsilon_a(\sin\alpha_{k_a})(\sin\beta_{k_a} - \cos\beta_{k_a})]\\
&\quad = 2^{-1}\sigma_{k_a}[\exp(-i\epsilon_a\alpha_{k_a})]|\cos\beta_{k_a}| + 2^{-1}\tau_{k_a}[\exp(i\epsilon_a\alpha_{k_a})|\sin\beta_{k_a}|
\end{aligned} \tag{18}$$

which is positive definite on $(U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2)^N$. But we have already seen that the Boltzmann factor is positive definite, so that the lemma just states that the integral of a (product of) positive definite function is positive.

Proof of the Theorem. The first inequalities in (i) and (ii) are obtained by summing (17) over appropriate choices of $(\epsilon_a)_{a\in A}$. We begin with (i), which, in duplicated form, is nothing but

$$\sum_{\substack{(\varepsilon_a)_{a\in A}\\ \sum_{a\in A}\varepsilon_a = 4 \bmod 8}} \left\langle\prod_{a\in A} (e^{i\epsilon_a\pi/4}S_{k_a}^1 + e^{-i\epsilon_a\pi/4}S_{k_a}'^1)\right\rangle \geqslant 0 \tag{19}$$

Indeed, if we expand the product using definition (16), we obtain

$$\sum_{\substack{(\varepsilon_a)_{a\in A}\\ \sum\varepsilon_a=4\,\mathrm{mod}\,8}} \Big[-\langle S_A{}^1\rangle - \langle S_A'^1\rangle + \sum_{\substack{B\subset A\\ B\neq\varnothing, A}} \left\{\exp\left[i\frac{\pi}{4}\left(\sum_{b\in B}\epsilon_b - \sum_{b'\in\tilde B}\epsilon_{b'}\right)\right]\right\}\langle S_B{}^1\rangle\langle S_C'^1\rangle\Big] \geqslant 0 \tag{20}$$

Let us compute the numerical factors separately:

$$\begin{aligned}
&\sum_{\substack{(\varepsilon_a)_{a\in A}\\ \sum\varepsilon_a=4\,\mathrm{mod}\,8}} \exp\left[i\frac{\pi}{4}\left(\sum_{b\in B}\epsilon_b - \sum_{b'\in\tilde B}\epsilon_{b'}\right)\right]\\
&= 8^{-1}\sum_{(\varepsilon_a)_{a\in A}}\sum_{q=0}^{7}\exp\left[i\frac{\pi}{4}\left(\sum_A\epsilon_a - 4\right)q\right]\exp\left[i\frac{\pi}{4}\left(\sum_B\epsilon_b - \sum_{\tilde B}\epsilon_{b'}\right)\right]\\
&= 8^{-1}\sum_{q=0}^{7}(-)^q\left\{\exp\left[i\frac{\pi}{4}(q+1)\right] + \exp\left[-i\frac{\pi}{4}(q+1)\right]\right\}^{|B|}\\
&\quad\times\left\{\exp\left[i\frac{\pi}{4}(q-1)\right] + \exp\left[-i\frac{\pi}{4}(q-1)\right]\right\}^{|\tilde B|}\\
&= \begin{cases} 2^{n-1} - 2^{2n-2} & \text{if } |B| = 2n \text{ or } |B| = 0\\ 0 & \text{if } |B| \text{ odd}\\ 2^{n-1} & \text{if } |B| \text{ even}, \quad |B|\neq 0, 2n\end{cases}
\end{aligned}\tag{21}$$

Inserting this result in (20) gives

$$-(2^{n-1}-1)(\langle S_A{}^1\rangle + \langle S_A'^1\rangle) \sum_{\substack{B\subset A\\ |B|\,\mathrm{even},\,\neq 0,2n}} \langle S_B{}^1\rangle\langle S_{\tilde B}'^1\rangle \geqslant 0$$

or

$$\langle S_A{}^1\rangle + \overline{\langle S_A{}^1\rangle} \leqslant (2^n-2)^{-1} \sum_{\substack{B\subset A\\ |B|\,\mathrm{even},\,\neq 0,2n}} (\langle S_B{}^1\rangle\overline{\langle S_{\tilde B}{}^1\rangle} + \overline{\langle S_B{}^1\rangle}\langle S_{\tilde B}{}^1\rangle) \tag{22}$$

If we now suppose that $(\mu_j)_j$ is real, (22) becomes the first inequality in (14). Similarly, to prove part (ii) of the theorem, we start from the lemma in the following combination:

$$\sum_{\substack{\sum_{A1\cup A2}\varepsilon_a=4\,\mathrm{mod}\,8\\ \sum_{A1}\varepsilon_{a1}-\sum_{A2}\varepsilon_{a2}=0\,\mathrm{mod}\,8}} \left\langle \prod_{a\in A_1\cup A_2}(e^{i\epsilon_a\pi/4}S_{k_a}^1 + e^{-i\epsilon_a\pi/4}S_{k_a}'^1)\right\rangle \geqslant 0 \tag{23}$$

Again, we expand the product

$$\sum_{\substack{\sum_{A1\cup A2}\varepsilon_a=4\,\mathrm{mod}\,8\\ \sum_{A1}\varepsilon_{a1}-\sum_{A2}\varepsilon_{a2}=0\,\mathrm{mod}\,8}} \sum_{\substack{B_1\subset A_1\\ B_2\subset A_2}} \left\{\exp\left[i\frac{\pi}{4}\left(\sum_{B_1\cup B_2}\epsilon_b - \sum_{\tilde B_1\cup\tilde B_2}\epsilon_{b'}\right)\right]\right\}$$

$$\times\ \langle S_{B_1}^1 S_{B_2}^1\rangle\langle S_{\tilde B_1}'^1 S_{\tilde B_2}'^1\rangle \geqslant 0 \tag{24}$$

and compute the numerical factors separately:

$$\sum_{\substack{\sum_{A_1\cup A_2} \epsilon_a = 4 \bmod 8 \\ \sum_{A_1} \epsilon_{a_1} - \sum_{A_2} \epsilon_{a_2} = 0 \bmod 8}} \exp\left[i\frac{\pi}{4}\left(\sum_{B_1\cup B_2} \epsilon_b - \sum_{\tilde{B}_1\cup\tilde{B}_2} \epsilon_{b'}\right)\right]$$

$$= 8^{-2}\sum_{\epsilon_a}\sum_{p,q=0}^{7} \exp\left[i\frac{\pi}{4}q\left(\sum_{A_1\cup A_2}\epsilon_a - 4\right)\right]$$

$$\times \exp\left[i\frac{\pi}{4}p\left(\sum_{A_1}\epsilon_{a_1} - \sum_{A_2}\epsilon_{a_2}\right]\right)$$

$$\times \exp\left[i\frac{\pi}{4}\left(\sum_{B_1\cup B_2}\epsilon_b - \sum_{\tilde{B}_1\cup\tilde{B}_2}\epsilon_{b'}\right)\right]$$

$$= 8^{-2}2\left(\sum_{q+p=0,2,4,6}\left\{i^{q+p}\exp\left[i\frac{\pi}{4}(q+p+1)\right] + \text{c.c.}\right\}^{|B_1|}\right.$$

$$\left.\times\left\{\exp\left[i\frac{\pi}{4}(q+p-1)\right] + \text{c.c.}\right\}^{|\tilde{B}_1|}\right)$$

$$\times\left(\sum_{q-p=0,2,4,6} i^{q-p}\left\{\exp\left[i\frac{\pi}{4}(q-p+1)\right] + \text{c.c.}\right\}^{|B_2|}\right.$$

$$\left.\times\left\{\exp\left[i\frac{\pi}{4}(q-p-1)\right] + \text{c.c.}\right\}^{|\tilde{B}_2|}\right)$$

$$+ 8^{-2}2\left\{\sum_{q+p=1,3,5,7}\cdots\right\}\left\{\sum_{q-p=1,3,5,7}\cdots\right\}$$

$$= \begin{cases} -2^{2n_1+2n_2-3} & \text{if } |B_1| = |B_2| = 0 \text{ or } |\tilde{B}_1| = |\tilde{B}_2| = 0 \\ 2^{2n_1+2n_2-3} & \text{if } |B_1| = |\tilde{B}_2| = 0 \text{ or } |\tilde{B}_2| = |B_2| = 0 \\ 2^{n_1+n_2-1} & \text{if } |B_1|, |B_2| \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Inserting this result in (24) gives the desired inequality in complex form (under the hypotheses of Theorem 2):

$$\langle S^1_{A_1} S^1_{A_2}\rangle + \text{c.c.} - (\langle S^1_{A_1}\rangle\overline{\langle S^1_{A_2}\rangle} + \text{c.c.})$$

$$\leqslant 2^{-n_1-n_2+1}\sum_{\substack{B_1\subset A_1; B_2\subset A_2 \\ |B_1|,|B_2| \text{odd}}} (\langle S^1_{B_1} S^1_{B_2}\rangle\overline{\langle S^1_{\tilde{B}_1} S^1_{\tilde{B}_2}\rangle} + \text{c.c.}) \tag{26}$$

The next theorem deals with odd correlation functions, in the presence of a "positive" external field. The inequalities will be *weaker* than the desired Gaussian inequalities.

Theorem 4. (Same hypotheses as Theorem 3.)

(iii) if $|A| = 2n + 1$,

$$\langle S_A{}^1\rangle \leqslant (2^{n+1} - 2)^{-1} \sum_{\substack{B \subset A \\ B = \varnothing, A}} \langle S_B{}^1\rangle\langle S_C{}^1\rangle$$

$$\leqslant \sum_{a_0 \in A} \sum_{\pi \in \mathscr{P}(A\setminus\{a_0\})} \langle S^1_{k_{a_0}}\rangle \prod_{\{j_1, j_2\}\in\pi} \langle S^1_{j_1} S^1_{j_2}\rangle \tag{27}$$

(iv) If $|A_1| = 2n_1 + 1$, $|A_2| = 2n_2$,

$$\langle S^1_{A_1} S^1_{A_2}\rangle - \langle S^1_{A_1}\rangle\langle S^1_{A_2}\rangle$$

$$\leqslant 2^{-n_1-n_2} \sum_{\substack{B_1 \subset A_1; B_2 \subset A_2 \\ |B_2| \text{odd}}} \langle S^1_{B_1} S^1_{B_2}\rangle\langle S^1_{\bar{B}1} S^1_{\bar{B}2}\rangle$$

$$\leqslant \sum_{a_0 \in A_1 \cup A_2} \sum_{\pi \in \mathscr{P}(A_1\setminus\{a_0\}; A_2\setminus\{a_0\})} \langle S^1_{k_{a_0}}\rangle \prod_{\{j_1, j_2\}\in\pi} \langle S^1_{j_1} S^1_{j_2}\rangle \tag{28}$$

Remark. An example for (27) is

$$\langle S_j{}^1 S_k{}^1 S_l{}^1\rangle \leqslant \langle S_j{}^1\rangle\langle S_k{}^1 S_l{}^1\rangle + \langle S_k{}^1\rangle\langle S_j{}^1 S_l{}^1\rangle + \langle S_l{}^1\rangle\langle S_j{}^1 S_k{}^1\rangle$$

whereas the relevant Gaussian inequality would be the GHS inequality, which has been proven for Ising-like systems:

$$\langle\sigma_j\sigma_k\sigma_l\rangle \leqslant \langle\sigma_j\rangle\langle\sigma_k\sigma_l\rangle + \langle\sigma_k\rangle\langle\sigma_j\sigma_l\rangle + \langle\sigma_l\rangle\langle\sigma_j\sigma_k\rangle - 2\langle\sigma_j\rangle\langle\sigma_k\rangle\langle\sigma_l\rangle$$

So far as we know, the GHS inequality is the only proven *odd* Gaussian inequality.

Proof of (iii) and (iv). We omit the calculations, which go like those in the proof of Theorem 3. The respective starting points are

$$\sum_{\substack{(\epsilon_a)_{a\in A} \\ \sum_{a\in A} \epsilon_a = \pm 3 \bmod 8}} \left\langle \prod_{a\in A} (e^{i\epsilon_a \pi/4} S^1_{k_a} + e^{-i\epsilon_a\pi/4} S'^1_{k_a}) \right\rangle \geqslant 0 \tag{29}$$

$$\sum_{\substack{(\epsilon_a)_{a\in A_1\cup A_2} \\ \sum_{a\in A_1\cup A_2} \epsilon_a = \pm 3 \bmod 8 \\ \sum_{a_1\in A_1} \epsilon_{a_1} - \sum_{a_2 \in A_2} \epsilon_{a_2} = \pm 1 \bmod 8}} \left\langle \prod_{a\in A_1\cup B_2} (e^{i\epsilon_a \pi/4} S^1_{k_a} + e^{i\epsilon_a\pi/4} S'^1_{k_a}) \right\rangle \geqslant 0 \tag{30}$$

As already mentioned, one knows how to bound correlation functions involving both components in terms of the correlation functions of the first component, which in turn are controlled by Theorem 3. To illustrate the present method, we shall, however, write down some inequalities for mixed components which we can obtain directly.

We look back at (8) and (9), where the direction 1 corresponds to $\beta = \pi/4$. We see that a cone of positive-definite functions on $U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by

$$e^{i\epsilon\pi/4}S_j^{\,1} + \eta e^{-i\epsilon\pi/4}S_j^{\,2} + e^{-i\epsilon\pi/4}S_j'^{1} + \eta e^{i\epsilon\pi/4}S_j'^{2} \tag{31}$$

with $\epsilon = \pm 1$, $\eta = \pm 1$.

Therefore Lemma 1 can be strengthened, with obvious notations and the same hypotheses, to the following:

Lemma 2. For any family of elements in $\{1,\ldots, N\}$ indexed by a finite set A, and for any function (ϵ, η) from A to $\{+1, -1\} \times \{+1, -1\}$, we have

$$\left\langle \prod_{a\in A} (e^{i\epsilon_a\pi/4}S^1_{k_a} + \eta_a e^{-i\epsilon_a\pi/4}S^2_{k_a} + e^{-i\epsilon_a\pi/4}S'^1_{k_a} + \eta_a e^{i\epsilon_a\pi/4}S'^2_{k_a}) \right\rangle \geqslant 0 \tag{32}$$

Corollary. For any function ϵ from A to $\{+1, -1\}$ and for any $A_2 \subset A$, we have

$$\left\langle \prod_{a\in A\setminus A_2} (e^{i\epsilon_a\pi/4}S^1_{k_a} + e^{-i\epsilon_a\pi/4}S'^1_{k_a}) \prod_{a\in A_2} (e^{-i\epsilon_a\pi/4}S^2_{k_a} + e^{i\epsilon_a\pi/4}S'^2_{k_a}) \right\rangle$$

$$\leqslant \left\langle \prod_{a\in A} (e^{i\epsilon_a\pi/4}S^1_{k_a} + e^{-i\epsilon_a\pi/4}S'^1_{k_a}) \right\rangle \tag{33}$$

Proof of the Corollary. Expand (32) in $A = A_1 \cup A_2$ and sum over η subject to $\eta_{A_2} \equiv \prod_{a\in A_2} \eta_a = \text{cst}$ for the given A_2.

The corollary allows us to strengthen Theorem 3 to the following:

Theorem 5. Let $(J^1_{jk}, J^2_{jk})_{j,k=1,\ldots,N}$ and $(\mu_j)_{j=1,\ldots,N}$ be as in Theorem 2. Suppose in addition that $(\mu_j)_j$ is real. Let $A = A_1 \cup A_2$, $\varnothing = A_1 \cap A_2$, $|A| = 2n$. Then:

(i)′ $$\Big|-\langle S^1_{A_1}\rangle\langle S^2_{A_2}\rangle + (2^n - 2)^{-1} \sum_{\substack{B_1\subset A_1; B_2\subset A_2\\ |B_1\cup\tilde{B}_2|\,\text{even},\, \neq 0,2n}} \langle S^1_{B_1}S^2_{B_2}\rangle\langle S^1_{\tilde{B}_1}S^2_{\tilde{B}_2}\rangle\Big|$$

$$\leqslant -\langle S_A{}^1\rangle + (2^n - 2)^{-1} \sum_{\substack{B\subset A\\ |B|\,\text{even},\, \neq 0,2n}} \langle S_B{}^1\rangle\langle S_{\tilde{B}}{}^1\rangle$$

(i″) $$\Big|-\langle S^1_{A_1}S^2_{A_2}\rangle + (2^n - 2)^{-1} \sum_{\substack{B_1\subset A_1; B_2\subset A_2\\ |B_1\cup B_2|\,\text{even},\, \neq 0,2n}} \langle S^1_{B_1}S^2_{B_2}\rangle\langle S^1_{\tilde{B}_1}S^2_{\tilde{B}_2}\rangle\Big|$$

$$\leqslant -\langle S^1_{A_1}\rangle\langle S^1_{A_2}\rangle + (2^n - 2)^{-1} \sum_{\substack{B_1\subset A_1; B_2\subset A_2\\ |B_1\cup\tilde{B}_2|\,\text{even},\, \neq 0,2n}} \langle S^1_{B_1}S^1_{B_2}\rangle\langle S^1_{\tilde{B}_1}S^1_{\tilde{B}_2}\rangle$$

(ii′) If $|A_2|$ is also even,

$$|\langle S^1_{A_1} S^2_{A_2}\rangle - \langle S^1_{A_1}\rangle\langle S^2_{A_2}\rangle + 2^{-n+1} \sum_{\substack{B_1 \subset A_1; B_2 \subset A_2 \\ |B_1|, |B_2| \text{odd}}} \langle S^1_{B_1} S^2_{B_2}\rangle\langle S^1_{\tilde{B}_1} S^2_{\tilde{B}_2}\rangle|$$

$$\leqslant -\langle S_A{}^1\rangle + \langle S^1_{A_1}\rangle\langle S^1_{A_2}\rangle + 2^{-n+1} \sum_{\substack{B_1 \subset A_1; B_2 \subset A_2 \\ |B_1|, |B_2| \text{odd}}} \langle S^1_{B_1} S^1_{B_2}\rangle\langle S^1_{\tilde{B}_1} S^2_{\tilde{B}_2}\rangle$$

(ii″) if $|A_2|$ is also even,

$$|-\langle S^1_{A_1} S^2_{A_2}\rangle + \langle S^1_{A_1}\rangle\langle S^2_{A_2}\rangle + 2^{-n+1} \sum_{\substack{B_1 \subset A_1; B_2 \subset A_2 \\ |B_1|, |B_2| \text{odd}}} \langle S^1_{B_1} S^2_{B_2}\rangle\langle S^1_{\tilde{B}_1} S^2_{\tilde{B}_2}\rangle|$$

$$\leqslant \langle S_A{}^1\rangle - \langle S^1_{A_1}\rangle\langle S^2_{A_2}\rangle + 2^{-n+1} \sum_{\substack{B_1 \subset A_1; B_2 \subset A_2 \\ |B_1|, |B_2| \text{odd}}} \langle S^1_{B_1} S^2_{B_2}\rangle\langle S^1_{\tilde{B}1} S^2_{\tilde{B}2}\rangle$$

Proof. Given the corollary, the proof is identical to that of Theorem 3.

Remark. When $(\mu_j)_j = 0$, Theorem 5 bounds the deviation from Gaussian correlations for mixed components by an analogous deviation for the first component.

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